MFAT Exam

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Part I Problems and Solutions
Part II Review of Linear Algebra

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Question:

On a questionnaire, a respondent must choose 3 of the 5 questions presented. How many different combinations of 3 questions can the respondent possibly choose.

Review: From n items choose m items without order $\binom{n}{m}$.

Solution:

$$\binom{5}{3} = 10.$$

Question:

A function f has the property that, at every point (x, y) on the curve $y = f(x)$, the slope of the line tangent to the curve is equal to $2xy$. What type of function is f.

Review: Slope of the tangent line to a function is the derivative of this function. To find this function is to solve a differential equation.

Solution:

$$\frac{dy}{dx} = 2xy, \quad \frac{1}{y}dy = 2xdx \quad \int \frac{1}{y}dy = \int 2xdx.$$

$$\ln y = x^2 + c, \quad y = e^{x^2+c}.$$

$y = f(x)$ is an exponential function.

Question:

Let A and B be matric spaces and let $f : A \to B$. Suppose that whenever X is an open set in B, the set $\{a \in A : f(a) \notin X\}$ is closed in A. Is f injective? Continuous? A homeomorphism?

Review:

- *Injective* **means one-to-one, that is if $f(x_1) = f(x_2)$ then $x_1 = x_2$.**
- **A function f from one metric space (X, dX) to another metric space (Y, dY). Then f is** *continuous* **at the point c in X if for any positive real number δ, there exists a positive real number ϵ such that all x in X satisfying $dX(x, c) < \delta$ will also satisfy $dY(f(x), f(c)) < \epsilon$.**
- **A function $f : X \to Y$, where X and Y are topological spaces, is** *continuous* **if and only if for every open set $V \subseteq Y$, the inverse image $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open.**
- **A function f between two topological spaces X and Y is called a** *homeomorphism* **if it is a bijection (1-1 and onto); continuous; the inverse function f^{-1} is continuous (f is an open mapping).**

Solution: Continuous.

Question:

In the xy-plane, the line that is tangent to the graph of $y = x^2$ at the point $(2, 4)$ has a slope of?

Review: Slope to a function at a point is the derivative of the function evaluated at the x-coordinate of the function.

Solution:

$$y' = 2x|_{x=2} = 2(2) = 4.$$

Question:

The set $\{1, 2, 4, 7, 8, 11, 13, 14\}$ **forms a group under the operation of multiplication modulo 15. What is the cyclic subgroup generated by** $\{7\}$ **?**

Review: A group G **is called cyclic if there exists an element** $g \in G$ **such that** $G =< g >= \{g^n | n \in \mathbb{Z}\}$**. Since any group generated by an element in a group is a subgroup of that group, showing that the only subgroup of a group** G **that contains** g **is** G **itself suffices to show that** G **is cyclic.**

Solution: { 1, 7, 4, 13} Under modulo 15, we obtain the following

$$7^0 = 1,\ 7^1 = 7,\ \ 7^2 = 49 \equiv 4,$$

$$7^5 = 4 * 7 = 28 \equiv 13,\ \ 7^6 = 13 * 7 = 91 \equiv 1.$$

Question:

A function $f(x)$ **is such a function where** $f(0) = 0$ **and** $f(1/n) = 0$ **for each positive integer** n**. If the graph if** $f(x)$ **between** $x = \dfrac{1}{n+1}$ **and** $x = \dfrac{1}{x}$ **consists of the congruent sides of an isosceles triangles of height 1 for each positive integer** n**. Then** $\int_0^1 f(x)dx = ?$

Review: To find the integral of a function in a given interval is to find the area bounded by the function and x**-axis in the given interval .**

Solution:

$$\int_0^1 f(x)dx = \sum_{i=1}^{\infty} \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1})$$

$$= 1/2((1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \cdots = 1/2.$$

Question:

A function $f(x)$ is differentiable on the interval $0 < x < 4$ if $f(1) = 1$, $f(3) = 7$, then for some $1 < c < 3$, $f'(c) =?$

Review: *Mean Value Theorem* **Let $f : [a, b] \to R$ be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b). Then, there exists some $c \in (a, b)$ such that**

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The mean value theorem is a generalization of Rolle's theorem, which assumes $f(a) = f(b)$, so that the right-hand side above is zero.

A continuous function need not be differentiable, but differentiable implies continues.

Solution:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = (7 - 1)/(3 - 1) = 3.$$

Question:

If $a, b \in \mathbb{Z}$, how many matrices of the form $\begin{bmatrix} 2 & a \\ b & 3 \end{bmatrix}$ are not invertible?

Review: For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we must have that

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

A matrix is invertible, then the determinant of the matrix cannot be zero.

Solution:

$$ab \neq 6.$$

The possibility that $ab = 6$ is all the possible factors

$$(\pm 1, \pm 6), \;\; (\pm 6, \pm 1), \;\; (\pm 2, \pm 3), \;\; (\pm 3, \pm 2).$$

Question:

If V_n is the real vector space of all n-tuples of real numbers for each $n > 1$, which of the following is true

1. **every basis of V_n contains exactly n vectors.**
2. **every basis of V_n is an orthogonal set of vectors.**
3. **every set of $n+1$ vectors of V_n is a linearly dependent set.**

Review: A *basis* B of a vector space V is a linearly independent subset of V that spans (or generates) V.

Suppose that $B = \{v_1, \ldots, v_n\}$ is a finite subset of a vector space V over a field F (such as the real or complex numbers $\mathbb{R}$ or $\mathbb{C}$). Then B is a basis if it satisfies: the linear independence property, that is

$$\forall a_1, \ldots, a_n \in F, \;\; a_1 v_1 + \cdots + a_n v_n = 0, \Rightarrow a_1 = \ldots = a_n = 0;$$

and the spanning property, that is

$$\forall x \in V, \exists a_1, \ldots, a_n \in F, \text{ such that } x = a_1 v_1 + \cdots + a_n v_n.$$

A subset $\{v_1, ..., v_k\}$ of a vector space V, with the inner product $<,>$, is called orthonormal if $< v_i, v_j >= 0$ when $i \neq j$. That is, the vectors are mutually perpendicular. Moreover, they are all required to have length one: $< v_i, v_i >= 1$.

Solution: Item 1 and 3 are true.

Question:

If $F(x) = \int_e^x \log t dt$ **for all positive** x**, then** $F'(x) =?$

Review: *Fundamental Theorem of Calculus* **There are two parts to the Fundamental Theorem of Calculus.**

Part I. Let f **be a continuous real-valued function defined on a closed interval** $[a, b]$**. Let** F **be the function defined, for all** $x \in [a, b]$**, by** $F(x) = \int_a^x f(t)dt$**. Then,** F **is differentiable on** $[a, b]$**, and for every** $x \in [a, b]$**,**

$$F'(x) = f(x).$$

Part II. Let f **be a continuous real-valued function defined on a closed interval** $[a, b]$**. Let** F **be an antiderivative of** f**, that is one of the infinitely many functions such that, for all** $x \in [a, b]$**,** $f(x) = F'(x)$**. Then**

$$\int_a^b f(x)dx = F(b) - F(a).$$

Solution: By Part I.

$$F'(x) = \log x.$$

Question:

If $F(1) = 2$ **and** $F(n) = F(n-1) + 1/2$ **for all integers** $n > 1$**, then** $F(101) = ?$

Review: *An arithmetic series* **is the sum of a sequence** $\{a_k\},\ \ k = 1, 2, \ldots$ **in which each term is computed from the previous one by adding (or subtracting) a constant** d**. Therefore, for**

$$k > 1,\ \ a_k = a_{(k-1)} + d = a_{(k-2)} + 2d = ... = a_1 + d(k-1).$$

The sum of the sequence of the first n **terms is then given by**

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} [a_1 + (k-1)d]$$

$$a_1 + a_n = a_1 + [a_1 + d(n-1)] = 2a_1 + d(n-1)$$

$$S_n = 1/2n(a_1 + a_n)$$

Solution:

$$F(101) = 2 + (101-1)(1/2) = 52.$$

Question:

If $b > 1$ **and if** $\int_0^b x dx = \int_0^b x^2 dx$**, then** $\int_1^b (x^2 - x) dx = ?$

Review: To find the definite integral, consider the regions bounded by certain curves.

Solution: First, find the places where the functions intersect each other.

$$x = x^2 \Rightarrow x = 0 \text{ or } x = 1.$$

Second, that $\int_0^b x dx = \int_0^b x^2 dx$ implies that the area bounded by two functions between the interval $[0, 1]$ and $[1, b]$ are equal,

$$\int_1^b (x^2 - x) dx = \int_0^1 (x - x^2) dx = [(x/2)^2 - (x^3)/3]_0^1 = 1/6.$$

Question:

What is the graph of $\{(\sin t,\ \cos t) \mid -\pi/2 \leq t \leq 0.\}$ **in the** xy**-plane.**

Review: Note $\sin^2 t + \cos^2 t = 1$ **therefore, the graph is a part of a unit circle. Also, need to know the value** $-1 \leq \sin t \leq 0$ **and** $0 \leq \cos t \leq 1$ **for** $-\pi/2 \leq t \leq 0$ **.**

Solution: It is part of circle in the 2nd quadrant.

Question:

$\int_0^1 \dfrac{x}{1+x^2}dx = ?$

Review: Find definite integral, if the regular formula does not apply try substitution first, then think about other integral techniques.

Solution:

$$t = 1 + x^2, \;\; dt = 2xdx, \;\; dx = \frac{dt}{2x},$$

$$x = 0 \Rightarrow t = 1, \;\; x = 1 \Rightarrow t = 2,$$

$$\int_0^1 \frac{x}{1+x^2}dx = \int_1^2 \frac{x}{t}\frac{dt}{2x} = \frac{1}{2}\int_1^2 \frac{1}{t}dt = \ln|t|\,|_1^2 = \frac{1}{2}\ln 2 = \ln\sqrt{2}.$$

Question:

If S is a nonempty finite set with k elements, then what is the number of one-to-one functions from S onto S?

Review: A *bijection*, **or a bijective function is a function f from a set X to a set Y with the property that, for every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.**

Alternatively, f is bijective if it is a one-to-one correspondence between those sets; i.e., both one-to-one (injective) and onto (surjective).

A bijective function is also called a permutation. This is more commonly used when $X = Y$.

Solution:

$$k!$$

Question:

For all real numbers x, y, the expression $\frac{x+y+|x-y|}{2}$ **is equal to which of the following**

1. **the maximum of x and y**
2. **the minimum of x and y**
3. $x+y$
4. **the average of x and y**
5. **the average of $|x+y|$ and $x-y$**

Review: $|X| = \begin{cases} x & \textbf{if } x \geq 0 \\ -x & \textbf{if } x \leq 0. \end{cases}$

Solution:

$$\frac{x+y+|x-y|}{2} = \begin{cases} \frac{x+y+x-y}{2} = x & \text{if } x > y \\ \frac{x+y+(-(x-y))}{2} = y & \text{if } x < y \end{cases}$$

Item 1 is true.

Question:

A drawer contains 2 blue, 4 red, 2 yellow socks. If 2 socks are to be randomly selected from the drawer, what is the prbability that they will be the same color?

Review: A probability of an event, A is represented by a real number in the range from 0 to 1 and written as $P(A)$. An impossible event has a probability of 0, and a certain event has a probability of 1.

The chance of the opposite of an event $P(notA) = 1 - P(A)$. If two events, A and B are independent then the joint probability is

$$P(A \textbf{ and } B) = P(A \cap B) = P(A)P(B).$$

For example if two coins are flipped the chance of both being heads is

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

If two events are mutually exclusive then the probability of either occurring is

$$P(A \textbf{ or } B) = P(A \cup B) = P(A) + P(B).$$

For example, the chance of rolling a 1 or 2 on a six-sided die is

$$P(1 \textbf{ or } 2) = P(1) + P(2) = \tfrac{1}{6} + \tfrac{1}{6} = \tfrac{1}{3}.$$

If the events are not mutually exclusive then

$$P\,(A \textbf{ or } B) = P\,(A) + P\,(B) - P\,(A \textbf{ and } B)\,.$$

Conditional probability is the probability of some event A, given the occurrence of some other event B. Conditional probability is written $P(A|B)$, and is read "the probability of A, given B". It is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

If $P(B) = 0$ then $P(A \mid B)$ is undefined.

Solution:

$$\frac{\binom{2}{2} + \binom{4}{2} + \binom{2}{2}}{\binom{2+4+2}{2}} = 2/7.$$

Question:

For what value of m is the vector $(1, 2, m, 5)$ a linear combination of the vectors $(0, 1, 1, 1)$, $(0, 0, 0, 1)$, $(1, 1, 2, 0)$?

Review: Suppose that K is a field and V is a vector space over K. As usual, we call elements of V vectors and call elements of K scalars. If $v_1, \ldots, v_n$ are vectors and $a_1, \ldots, a_n$ are scalars, then the linear combination of those vectors with those scalars as coefficients is

$$a_1v_1 + a_2v_2 + a_3v_3 + \cdots + a_nv_n.$$

Solution:

$$(1, 2, m, 5) = a(0, 1, 1, 1) + b(0, 0, 0, 1) + c(1, 1, 2, 0)$$

$$1 = c, \;\; 2 = a + c, \;\; m = a + 2c, \;\; 5 = a + b$$

$$c = 1, \;\; a = 1, \;\; b = 4, \;\; m = 3.$$

Question:

Which of the following statements are true for every function f, defined on the set of all real numbers such that $\lim_{x\to 0}\dfrac{f(x)}{x}$ is a real number L and $f(0)=0$.

1. f **is differentiable**
2. $L=0$
3. $\lim_{x\to 0}\dfrac{f(x)}{x}=0$

Review: A real function is said to be *differentiable* **at a point if its derivative exists at that point. That is**

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} \textbf{ exists.}$$

A function is continuous at $x=a$, if $\lim_{x\to a}f(x)$ exists and $\lim_{x\to a}f(x)=f(a)$.

A function is differentiable at a point, then this function must be continuous at this point.

Solution:

$\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}=\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\dfrac{f(x)}{x}$ is a real number. So function is differentiable. Also, the function is differentiable at $x=0$ implies it is continuous at $x=0$, that is $\lim_{x\to 0}f(x)=f(0)=0$.

So Item 1 and 3 are correct.

Question:

Let $\mathbb{Z}$ be the group of all integers under the operation of addition. Which of the following is not a group:

1. $\{0\}$
2. $\{n \in \mathbb{Z} \mid n \geq 0\}$
3. $\{n \in \mathbb{Z}`\mid n$ **is even** $\}$
4. $\{n \in \mathbb{Z}`\mid n$ **is divisible by** 6 **and** $9\}$

Review: A *group* $(G, *)$ **is a set** G **with a binary operation** $*$ **that satisfies the following four axioms:**

- **Closure:** $\forall a, b \in G \Rightarrow a * b \in G.$
- **Associativity:** $\forall a, b, c \in G \Rightarrow (a * b) * c = a * (b * c).$
- **Identity element: There exists an element** $e \in G$ **such that** $\forall a \in G, e * a = a * e = a$**.**
- **Inverse element: For each** $a \in G$**, there exists an element** $b \in G$ **such that** $a * b = b * a = e$**, where** e **is an identity element.**

An *additive group* **is a group where the operation is called addition and is denoted** $+$**. In an additive group, the identity element is called zero, and the inverse of the element** a **is denoted** $-a$ **(minus a).**

Solution: Since we must have an inverse element in the group, Item 2 is not a group.

Review on Linear Algebra

The matrix

$$\mathbf{A} = \begin{bmatrix} 9 & 8 & 6 \\ 1 & 2 & 7 \\ 4 & 9 & 2 \\ 6 & 0 & 5 \end{bmatrix} or \mathbf{A} = \begin{pmatrix} 9 & 8 & 6 \\ 1 & 2 & 7 \\ 4 & 9 & 2 \\ 6 & 0 & 5 \end{pmatrix}$$

is a 4×3 matrix. The element $a_{2,3}$ or $\mathbf{A}[2,3]$ is 7.

The matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix}$$

is a 1×9 matrix, or 9-element row vector.

Matrix addition

Two or more matrices of identical dimensions m and n can be added. Given m-by-n matrices A and B, their sum $A + B$ is the m-by-n matrix computed by adding corresponding elements:

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (a_{i,j})_{1 \le i \le m;\, 1 \le j \le n} + (b_{i,j})_{1 \le i \le m;\, 1 \le j \le n} \\ &= (a_{i,j} + b_{i,j})_{1 \le i \le m; 1 \le j \le n} \end{aligned}$$

For example:

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 7 & 5 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 & 1+5 \\ 1+7 & 0+5 & 0+0 \\ 1+2 & 2+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 8 & 5 & 0 \\ 3 & 3 & 3 \end{bmatrix}$$

Another, much less often used notion of matrix addition is the direct sum.

Scalar multiplication

Given a matrix A and a number c, the scalar multiplication cA is computed by multiplying every element of A by the scalar c (i.e. $(c\mathbf{A})_{i,j} = c \cdot a_{i,j}$). For example:

$$2 \cdot \begin{bmatrix} 1 & 8 & -3 \\ 4 & -2 & 5 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 & 2 \cdot 8 & 2 \cdot -3 \\ 2 \cdot 4 & 2 \cdot -2 & 2 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 16 & -6 \\ 8 & -4 & 10 \end{bmatrix}$$

Matrix addition and scalar multiplication turn the set $\mathrm{M}(m, n, \mathbb{R})$ of all m-by-n matrices with real entries into a real vector space of dimension $m \cdot n$.

Matrix multiplication

Multiplication of two matrices is well-defined only if the number of columns of the left matrix is the same as the number of rows of the right matrix. The middle dot $(\cdot)$ is not used to indicate matrix multiplication (it is used for scalar multiplication). If A is an m-by-n matrix and B is an n-by-p matrix, then their matrix product AB is the m-by-p matrix given by:

$$(\mathbf{AB})_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \ldots + a_{i,n}b_{n,j}$$

for each pair (i, j). For example:

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} (1 \times 3 + 0 \times 2 + 2 \times 1) & (1 \times 1 + 0 \times 1 + 2 \times 0) \\ (-1 \times 3 + 3 \times 2 + 1 \times 1) & (-1 \times 1 + 3 \times 1 + 1 \times 0) \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}.$$

Matrix multiplication has the following properties

I. $(AB)C = A(BC)$ for all k-by-m matrices A, m-by-n matrices B and n-by-p matrices C ("associativity").

II. $(A + B)C = AC + BC$ for all m-by-n matrices A and B and n-by-k matrices C ("right distributivity").

III. $C(A + B) = CA + CB$ for all m-by-n matrices A and B and k-by-m matrices C ("left distributivity").

Matrix multiplication is not commutative; that is, given matrices A and B and their product defined, then generally $AB \neq BA$. It may also happen that AB is defined but BA is not defined.

Linear transformations, ranks and transpose

Matrices can conveniently represent linear transformations because matrix multiplication neatly corresponds to the composition of maps.

For every linear map $f : R^n \to R^m$ there exists a unique m-by-n matrix A such that $f(x) = Ax$ for all $x \in R^n$.

Now if the k-by-m matrix B represents another linear map $g : R^m \to R^k$, then the linear map $g \circ f$ is represented by BA.

More generally, a linear map from an n-dimensional vector space to an m-dimensional vector space is represented by an m-by-n matrix, provided that bases have been chosen for each.

The **rank** of a matrix A is the dimension of the image of the linear map represented by A; this is the same as the dimension of the space generated by the rows of A, and also the same as the dimension of the space generated by the columns of A.

The **transpose** of an m-by-n matrix A is the n-by-m matrix A^t formed by turning rows into columns and columns into rows, i.e. $A^t[i, j] = A[j, i]$ for all indices i and j.

$$(A + B)^t = A^r + B^r, \ \ (AB)^t = B^t A^t.$$

Square matrices

A **square matrix** is a matrix which has the same number of rows and columns. The unit matrix or identity matrix I^n, with elements on the main diagonal set to 1 and all other elements set to 0, satisfies $MI^n = M$ and $I^nN = N$ for any m-by-n matrix M and n-by-k matrix N. For example, if $n = 3$:

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Invertible elements in this ring are called **invertible matrices** or non-singular matrices. An n by n matrix A is invertible if and only if there exists a matrix B such that

$$AB = I^n = BA.$$

In this case, B is the inverse matrix of A, denoted by A^{-1}. The set of all invertible n-by-n matrices forms a group under matrix multiplication, the general linear group.

Inversion of a 2 by 2 matrices can be done easily as follows

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Finding the inverse of a matrix

Suppose A is a $n \times n$ matrix and you need to calculate its inverse. The $n \times n$ identity matrix is augmented to the right of A, forming a $n \times 2n$ matrix (the block matrix $B = [A, I]$). Through application of elementary row operations and the Gaussian elimination algorithm, the left block of B can be reduced to the identity matrix I, which leaves A^{-1} in the right block of B.

If the algorithm is unable to reduce A to triangular form, then A is not invertible.

In practice, inverting a matrix is rarely required. Most of the time, one is really after the solution of a particular system of linear equations.

The **determinant** of a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\det(A) = ad - bc$.

Using the cofactor expansion on the first row of the matrix we get the **determinant** of a 3×3 matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

is

$$\begin{aligned} \det(A) &= a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= aei - afh - bdi + cdh + bfg - ceg \\ &= (aei + bfg + cdh) - (gec + hfa + idb), \end{aligned}$$

Example

$$A = \begin{bmatrix} -2 & 2 & -3 \\ -1 & 1 & 3 \\ 2 & 0 & -1 \end{bmatrix}.$$

Expand the determinant along a row or column. It is best to choose a row or column with many zeros, so we will expand along the second column:

$$\det(A) \;=\; (-1)^{1+2} \cdot 2 \cdot \det \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix} + (-1)^{2+2} \cdot 1 \cdot \det \begin{bmatrix} -2 & -3 \\ 2 & -1 \end{bmatrix}$$

$$= \; (-2)\cdot((-1)\cdot(-1)-2\cdot3)+1\cdot((-2)\cdot(-1)-2\cdot(-3)) \;=\; (-2)(-5)+8 = 18.$$

If λ is a number and v is a non-zero vector such that $Av = \lambda v$, then we call v an **eigenvector** of A and λ the associated **eigenvalue**.

The number λ is an eigenvalue of A if and only if $A - \lambda I^n$ is not invertible, which happens if and only if $p_A(\lambda) = 0$. Here $p_A(x)$ is the **characteristic polynomial** of A:

$$p(x) = \det(xI - A)$$

where I is the identity matrix of the same dimension as A.

Example

Suppose we want to compute the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

We have to compute the determinant of

$$tI - A = \begin{pmatrix} t-2 & -1 \\ 1 & t \end{pmatrix}$$

and this determinant is

$$(t-2)t - 1(-1) = t^2 - 2t + 1.$$

The latter is the characteristic polynomial of A.

This is a polynomial of degree n and has therefore n complex roots (counting multiple roots according to their multiplicity). In this sense, every square matrix has n complex eigenvalues.

The determinant of a square matrix A is the product of its n eigenvalues. Invertible matrices are precisely those matrices with nonzero determinant.

Linear Transformation Some special cases of linear transformations of two-dimensional space R^2 are illuminating:

I. rotation by 90 degrees counterclockwise:

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

II. reflection against the x axis:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

III. scaling by 2 in all directions:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

IV. vertical shear mapping:

$$A = \begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}$$

V. squeezing:

$$A = \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix}$$

VI. projection onto the y axis:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

In mathematics, a **system of linear equations** (or linear system) is a collection of linear equations involving the same set of variables.

Example

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \tfrac{1}{2}y - z &= 0 \end{aligned}$$

is a system of three equations in the three variables x, y, z. A solution to a linear system is an assignment of numbers to the variables such that all the equations are simultaneously satisfied. A solution to the system above is given by

$$\begin{aligned} x &= 1 \\ y &= -2 \\ z &= -2 \end{aligned}$$

since it makes all three equations valid.

A **general system of m linear equations with n unknowns** can be written as

$$\begin{aligned} a_{11}x_1 + \ & a_{12}x_2 + \cdots + \ a_{1n}x_n = b_1 \\ a_{21}x_1 + \ & a_{22}x_2 + \cdots + \ a_{2n}x_n = b_2 \\ \vdots \quad & \quad \vdots \qquad\qquad \vdots \qquad \vdots \\ a_{m1}x_1 + & a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

Here $x_1,\ x_2, ..., x_n$ are the unknowns, $a_{11},\ a_{12}, ...,\ a_{mn}$ are the coefficients of the system, and $b_1,\ b_2, ..., b_m$ are the constant terms.

Often the coefficients and unknowns are real or complex numbers, but integers and rational numbers are also seen, as are polynomials and elements of an abstract algebraic structure.

Vector equation

One extremely helpful view is that each unknown is a weight for a column vector in a linear combination.

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Matrix equation

The vector equation is equivalent to a matrix equation of the form

$$A\mathbf{x} = \mathbf{b}$$

where A is an $m \times n$ matrix, x is a column vector with n entries, and b is a column vector with m entries.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The number of vectors in a basis for the span is now expressed as the rank of the matrix.

Solve System of Linear Equations There are several algorithms for solving a system of linear equations.

Describe Solution: Some of the variables are designated as free (or independent, or as parameters), meaning that they are allowed to take any value, while the remaining variables are dependent on the values of the free variables.

Example

$$
\begin{aligned}
x + 3y - 2z &= 5 \\
3x + 5y + 6z &= 7
\end{aligned}
$$

The solution set to this system can be described by the following equations with z a free variable:

$$x = -7z - 1 \quad \text{and} \quad y = 3z + 2.$$

Elimination of variables

The simplest method for solving a system of linear equations is to repeatedly eliminate variables.

I. In the first equation, solve for the one of the variables in terms of the others.

II. Plug this expression into the remaining equations. This yields a system of equations with one fewer equation and one fewer unknown.

III. Continue until you have reduced the system to a single linear equation.

IV. Solve this equation, and then back-substitute until the entire solution is found.

Example

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

Solving the first equation for x gives $x = 5 + 2z3y$, and plugging this into the second and third equation yields

$$\begin{aligned} -4y + 12z &= -8 \\ -2y + \ \ 7z &= -2 \end{aligned}$$

Solving the first of these equations for y yields $y = 2 + 3z$, and plugging this into the third equation yields $z = 2$. We now have:

$$\begin{aligned} x &= 5 + 2z - 3y \\ y &= 2 + 3z \\ z &= 2 \end{aligned}$$

Substituting $z = 2$ into the second equation gives $y = 8$, and substituting $z = 2$ and $y = 8$ into the first equation yields $x = 15$. Therefore, the solution set is the single point $(x, y, z) = (15, 8, 2)$.

Row reduction: Gaussian elimination

In row reduction, the linear system is represented as an augmented matrix. This matrix is then modified using elementary row operations until it reaches reduced row echelon form. There are three types of elementary row operations:

- Type 1: Swap the positions of two rows.
- Type 2: Multiply a row by a nonzero scalar.
- Type 3: Add to one row a scalar multiple of another.

Because these operations are reversible, the augmented matrix produced always represents a linear system that is equivalent to the original.

Example: Gaussian elimination to the following matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right].$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The last matrix is in reduced row echelon form, and represents the system $x = 15, y = 8, z = 2$.

Cramer's rule Cramer's rule is an explicit formula for the solution of a system of linear equations, with each variable given by a quotient of two determinants.

Example

$$\begin{aligned} x + 3y - 2z &= 5 \\ 3x + 5y + 6z &= 7 \\ 2x + 4y + 3z &= 8 \end{aligned}$$

is given by

$$x = \frac{\begin{vmatrix} 5 & 3 & -2 \\ 7 & 5 & 6 \\ 8 & 4 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} 1 & 5 & -2 \\ 3 & 7 & 6 \\ 2 & 8 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 2 & 4 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{vmatrix}}$$

For each variable, the denominator is the determinant of the matrix of coefficients, while the numerator is the determinant of a matrix in which one column has been replaced by the vector of constant terms.

Homogeneous systems

A system of linear equations is homogeneous if all of the constant terms are zero:

$$\begin{aligned} a_{11}x_1 + \ & a_{12}x_2 + \cdots + \ a_{1n}x_n = 0 \\ a_{21}x_1 + \ & a_{22}x_2 + \cdots + \ a_{2n}x_n = 0 \\ \vdots \qquad & \ \ \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + \ & a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{aligned}$$

A homogeneous system is equivalent to a matrix equation of the form

$$A\mathbf{x} = \mathbf{0}$$

where A is an $m \times n$ matrix, x is a column vector with n entries, and 0 is the zero vector with m entries.

Every homogeneous system has at least one solution, known as the zero solution (or trivial solution), which is obtained by assigning the value of zero to each of the variables. The solution set has the following additional properties:

1. If u and v are two vectors representing solutions to a homogeneous system, then the vector sum u + v is also a solution to the system.
2. If u is a vector representing a solution to a homogeneous system, and r is any scalar, then ru is also a solution to the system.

Nonhomogeneous systems

There is a close relationship between the solutions to a linear system and the solutions to the corresponding homogeneous system:

$$A\mathbf{x} = \mathbf{b} \quad \text{and} \quad A\mathbf{x} = \mathbf{0}.$$

Specifically, if p is any specific solution to the linear system $Ax = b$, then the entire solution set can be described as

$$\{\mathbf{p} + \mathbf{v} : \mathbf{v} \text{ is any solution to } A\mathbf{x} = \mathbf{0}\}$$